

# Branching diffusion representation of quasi-linear elliptic PDEs and estimation using Monte Carlo method\*

Ankush Agarwal<sup>†</sup>      Julien Claisse<sup>‡</sup>

Centre de Mathématiques Appliquées (CMAP)  
École Polytechnique and CNRS  
Route de Saclay, 91128 Palaiseau Cedex, France

## Abstract

We study quasi-linear elliptic PDEs with polynomial non-linearity and provide a probabilistic representation of their solution using branching diffusion processes and automatic differentiation formulas. We provide explicit sufficient conditions on different parameters of the PDE under which our solution holds and validate our theoretical results with the help of different numerical examples.

**Key words:** partial differential equation, elliptic, semi-linear, quasi-linear, exit time, branching diffusion processes, Green function, integration by parts, Monte Carlo methods

**AMS subject classifications (2010):** 35J61, 35J62, 60H30, 60J85, 65C05

## 1 Introduction

In this paper, we are concerned with the following class of quasi-linear elliptic partial differential equations (PDEs): given a bounded domain  $\mathcal{O} \subset \mathbb{R}^d$ ,  $f : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $h : \partial\mathcal{O} \rightarrow \mathbb{R}$ ,

$$\mathcal{L}u + f(u, Du) = 0 \quad \text{in } \mathcal{O}, \quad u = h \quad \text{on } \partial\mathcal{O} \quad (1)$$

where  $\mathcal{L}$  is the infinitesimal generator of a diffusion process. This class of PDE arises naturally in a wide variety of domains including geometry, physics, chemistry, finance and natural sciences (see, *e.g.*, Badiale and Serra [2]). Our aim is to provide a new probabilistic representation of the solution which is well suited for numerical application, especially in high dimensions.

The classical probabilistic approach for quasi-linear PDE relies on the theory of backward stochastic differential equations (BSDEs) initiated by Pardoux and Peng [25]. We refer the reader to Section 4 of Pardoux [24] for a detailed exposition on the link between BSDEs and PDE (1). Over the past few years, numerical methods for BSDEs have been introduced by Bally and Pagès [3], Bouchard and Touzi [5] and Zhang [31] among several others. The classical approach is based on a discretization of the time horizon which reduces the problem to a backward iteration scheme where every step consists in estimating a conditional expectation. The available methods to estimate conditional expectations, such as non-linear regression, are computationally expensive and suffer from the curse of dimensionality.

Our probabilistic representation is based on the so-called *branching diffusion processes*. These processes describe the evolution of a population of independent and identical particles moving according to a diffusion process. They were first introduced by Skorokhod [27] and further studied in a more thorough and systematic way in a series of papers by Ikeda, Nagasawa and Watanabe [18, 19, 20]. In particular, these authors established a new probabilistic representation of semi-linear parabolic PDEs of the form

$$\partial_t u + \mathcal{L}u + \beta \left( \sum_{l \in \mathbb{N}} p_l u^l - u \right) = 0 \quad \text{in } [0, T) \times \mathbb{R}^d, \quad u(T, \cdot) = g \quad \text{in } \mathbb{R}^d, \quad (2)$$

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<sup>†</sup>Email: ankush.agarwal@polytechnique.edu. The author research is part of the Chair *Financial Risks* of the *Risk Foundation*.

<sup>‡</sup>Email: julien.claisse@polytechnique.edu. The author acknowledges the financial support of ERC 321111 Rofirm.

where  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\beta > 0$  and  $(p_l)_{l \in \mathbb{N}}$  is a probability mass function. More precisely, consider a branching diffusion process where each particle moves according to a diffusion process with generator  $\mathcal{L}$ . The particle dies at an exponentially distributed random time with parameter  $\beta$  and gives birth to  $k$  offsprings with  $(p_k)_{k \in \mathbb{N}}$ . Then, the solution of PDE (2) is given by  $\mathbb{E}[\prod_{i=1}^{N_T} g(X_T^i)]$  where  $N_T$  and  $(X_T^i)_{i=1, \dots, N_T}$  denote the number of particles and their positions at time  $T$  respectively. The special case  $p_2 = 1 - p_0$ , which corresponds to the celebrated Fisher–KPP equation, has been particularly well-studied, see, *e.g.*, McKean [23].

Surprisingly, the representation of semi-linear PDEs with branching diffusion processes did not attract much attention for numerical application before the recent developments in Henry-Labordère [14] and Henry-Labordère, Tan and Touzi [15]. The authors extended the probabilistic representation of PDE (2) to the case where  $(p_l)_{l \in \mathbb{N}}$  is replaced by an arbitrary real-valued sequence of functions. This result was further extended in Henry-Labordère *et al.* [17] to parabolic quasi-linear PDE of the form

$$\partial_t u + \mathcal{L}u + f(u, Du) = 0 \quad \text{in } [0, T) \times \mathbb{R}^d, \quad u(T, \cdot) = g \quad \text{in } \mathbb{R}^d, \quad (3)$$

where  $f$  is a  $(d+1)$ -variate power series. The main idea consists in using marked branching diffusion and to exploit the so-called Bismuth-Elworthy-Li formula to account for the non-linearity in the gradient. Under conditions of “small non-linearity” or small maturity, the authors show that their probabilistic representation provides a continuously differentiable viscosity solution of PDE (3). We also refer the reader to Henry-Labordère, Tan and Touzi [16] and to Bouchard, Tan and Zou [6] for further developments on numerical applications of branching diffusion processes.

The aim of this paper is to extend the result in Henry-Labordère, Tan and Touzi [15] and Henry-Labordère *et al.* [17] to the elliptic PDE case. The first results on the link between semi-linear elliptic PDE and branching diffusion processes were obtained by Watanabe [30] who gave a criterion for the extinction of branching diffusion processes absorbed at the boundary of a domain. Recently, Bossy *et al.* [4] extended this result to derive a probabilistic representation for PDE of the form

$$\mathcal{L}u + \beta \left( \sum_{l \in \mathbb{N}} c_l u^l - u \right) = 0 \quad \text{in } \mathcal{O}, \quad u = h \quad \text{on } \partial \mathcal{O}, \quad (4)$$

where  $(c_l)_{l \in \mathbb{N}}$  is a sequence of real-valued functions. They used it to compute a Monte Carlo approximation of the solution to the Poisson-Boltzmann equation. The critical assumption in their work is the existence of a smooth solution to PDE (4). We extend their analysis in this paper as we show directly that the probabilistic representation provides a continuous viscosity solution to PDE (4). The main difficulty compared to [15] is to ensure the integrability of the probabilistic representation. Indeed, the arguments of small maturity used in the aforementioned paper cannot be exploited here as the maturity is by nature infinite in our setting.

We also perform rigorous analysis for the case of quasi-linear elliptic PDE. To the best of our knowledge, this is the first paper in the literature to provide a representation for this class of PDE using branching diffusion processes. In contrast with Henry-Labordère *et al.* [17], Malliavin calculus cannot be used in this setting since the exit time of a diffusion from a domain is not differentiable w.r.t. the starting point. However Delarue [9] and Gobet [12] established a suitable automatic differentiation formula based on the work of Thalmaier [28, 29]. This allows us to derive a probabilistic representation analogous to [17] and to show that, under suitable conditions, it provides a continuously differentiable solution to PDE (1) when  $f$  is a multivariate power series.

The paper is organized as follows. In the next section, we provide a precise formulation of the problem and we introduce the branching diffusion processes used to derive our stochastic representation. Then we consider the semi-linear case in Section 3 and we give explicit sufficient condition to ensure that the probabilistic representation provides a continuous viscosity solution to the PDE. In Section 4, we extend the analysis to the quasi-linear case. Under suitable condition, it is shown that the probabilistic representation is a continuous differentiable viscosity solution to the PDE. Finally, we present some numerical examples to illustrate our results in the last section.

## 2 Framework

**2.1 Problem definition** Let for  $d \geq 1$ ,  $\mathbb{M}^d$  denote the set of all  $d \times d$  matrices, and  $(\mu, \sigma) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{M}^d$  denote the coefficient functions. Then for a non-negative integer  $m$ , we consider a subset  $L \subset \mathbb{N}^{m+1}$  and a generator function  $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  defined as

$$f(x, y, z) := \sum_{l=(l_0, l_1, \dots, l_m) \in L} c_l(x) y^{l_0} \prod_{i=1}^m (b_i(x) \cdot z)^{l_i},$$

where  $(c_l)_{l \in L}, c_l : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $(b_i)_{i=1, \dots, m}, b_i : \mathbb{R}^d \rightarrow \mathbb{R}$  are sequences of functions. In particular,  $c_l(x)y^{l_0} \prod_{i=1}^m (b_i(x) \cdot z)^{l_i} = c_l(x)y^{l_0}$  when  $l = (l_0, 0, \dots, 0)$ . Further, for every  $l = (l_0, l_1, \dots, l_m)$ , we denote  $|l| := \sum_{i=0}^m l_i$ . Given a bounded domain  $\mathcal{O} \subset \mathbb{R}^d$ , we consider the following quasi-linear elliptic PDE:

$$\begin{aligned} \mathcal{L}u + \beta(f(\cdot, u, Du) - u) &= 0, & \text{in } \mathcal{O}, \\ u &= h, & \text{on } \partial\mathcal{O}, \end{aligned} \quad (5)$$

where  $\beta$  is a positive constant,  $h : \partial\mathcal{O} \rightarrow \mathbb{R}$  is the Dirichlet boundary condition and  $\mathcal{L}$  is the infinitesimal generator associated to a diffusion process with coefficients  $(\mu, \sigma)$ . Under a set of general assumptions, we provide a probabilistic representation of the solution  $u$  using the theory of branching diffusion processes. We list the assumptions on parameters of PDE (5) at the outset which are needed for our results.

**Assumption 2.1.** (i) *The functions  $b_i, c_l$  are continuous on  $\bar{\mathcal{O}}$ .*

(ii) *The function  $h$  is continuous on  $\partial\mathcal{O}$ .*

(iii) *The function  $f$  is continuous on  $\bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^d$ .*

**2.2 Branching diffusion processes** A (age-dependent) marked branching diffusion process is characterized by the diffusion coefficient  $(\mu, \sigma)$ , a probability density function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , and a probability mass function  $(p_l)_{l \in L}$ . In this process, we start with one particle of mark 0 born at position  $x \in \mathbb{R}^d$  which undergoes a diffusion  $(\mu, \sigma)$  during its lifetime distributed according to  $\rho$ . At the end of its lifetime (arrival time), the particle dies and gives rise to  $|l|$  offsprings with probability  $p_l$ , among which  $l_0$  have mark 0,  $l_1$  have mark 1, and so on. After their birth, each offspring performs the same but an independent branching diffusion process as the parent particle. Additionally, we consider that particles die without offspring when they leave the domain  $\mathcal{O}$ . Typically,  $\rho$  is assumed to be an exponential distribution as it gives rise to a Markov process with a closed-form generating function. However, we do not put such a restriction. In order to construct the above process, we consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  equipped with

- a sequence of i.i.d. positive random variables  $(\tau^{n, \tilde{n}})_{n, \tilde{n} \geq 1}$  distributed with density function  $\rho$
- a sequence of i.i.d. random elements  $(I^{n, \tilde{n}})_{n, \tilde{n} \geq 1} \in L$  with  $\mathbb{P}(I^{n, \tilde{n}} = l) = p_l, l \in L$
- a sequence of independent  $d$ -dimensional Brownian motions  $(W^{n, \tilde{n}})_{n, \tilde{n} \geq 1}$

Further, we consider the sequences  $(\tau^{n, \tilde{n}})_{n, \tilde{n} \geq 1}, (W^{n, \tilde{n}})_{n, \tilde{n} \geq 1}$  and  $(I^{n, \tilde{n}})_{n, \tilde{n} \geq 1}$  to be mutually independent. The age-dependent branching process is constructed as follows:

1. Start from a particle at position  $x \in \mathbb{R}^d$  and index it by label (1), of generation 1. Denote  $(1)^- = \emptyset$ ,  $T_\emptyset = 0$  and  $X_\emptyset = x$ .
2. For generation  $n$ , let the label for a particle be given as  $k = (k_1, \dots, k_{n-1}, k_n) \in \mathbb{N}^n$ . Further, denote by  $k^- := (k_1, \dots, k_{n-2}, k_{n-1})$  the parent particle of  $k$  and  $\pi_n(k)$  as an injection from  $\mathbb{N}^n \rightarrow \mathbb{N}$ . The particle  $k$  starts from  $X_{T_{k^-}}^{k^-}$  at time  $T_{k^-}$ :

- The position  $X^k$  of the particle during its lifetime is given by

$$X_t^k = X_{T_{k^-}}^{k^-} + \int_{T_{k^-}}^t \mu(X_s^k) ds + \int_{T_{k^-}}^t \sigma(X_s^k) dW_s^k, \quad \mathbb{P} - \text{a.s.}$$

where the Brownian motion  $W^k$  is defined by

$$W_t^k := W_{T_{k^-}}^{k^-} + \Delta W_{t-T_{k^-}}^k, \quad \text{with } \Delta W_{t-T_{k^-}}^k := W_{t-T_{k^-}}^{n, \pi_n(k)}.$$

- The arrival time of the particle is given by

$$T_k := \left( T_{k^-} + \tau^{n, \pi_n(k)} \right) \wedge \inf \{ t \geq T_{k^-} \text{ s.t. } X_t^k \notin \mathcal{O} \}.$$

- At the arrival time, if  $X_{T_k}^k \notin \mathcal{O}$ , then the particle  $k$  dies without offsprings, else, given  $I^k = I^{n, \pi_n(k)}$ , the particle  $k$  dies and gives rise to  $|I^k|$  offsprings which constitutes the  $(n+1)$ th generation, and are indexed by label  $(k_1, \dots, k_{n-1}, k_n, i)$  for  $i = 1, \dots, |I^k|$ .
- When  $I^k = (\bar{l}_0, \bar{l}_1, \dots, \bar{l}_m)$ , we have  $|\bar{l}| := \sum_{i=0}^m \bar{l}_i$  offspring particles, among which the first  $\bar{l}_0$  have mark 0,  $\bar{l}_1$  have mark 1, and so on, so that each particle has mark  $i$  for  $i = 0, \dots, m$ .

In addition, we denote by  $\mathcal{K}^n$  the collection of particles of the  $n$ th generation and by  $\mathcal{K} = \bigcup_{n \geq 1} \mathcal{K}_n$  the collection of all particles in the branching diffusion. Finally, we introduce the  $\sigma$ -fields

$$\mathcal{F}_n := \sigma(\tau^{i,\tilde{n}}, I^{i,\tilde{n}}, W^{i,\tilde{n}}, 1 \leq i \leq n, \tilde{n} \geq 1).$$

We make the following assumptions to ensure that the branching diffusion process above is well-defined and for further developments.

**Assumption 2.2.** (i) *The probability distribution  $(p_l)_{l \in L}$  satisfies  $p_l > 0$  and  $\sum_{l \in L} lp_l < \infty$ .*  
(ii) *The probability density  $\rho$  is strictly positive.*  
(iii) *The coefficients  $(\mu, \sigma)$  are Lipschitz on  $\bar{\mathcal{O}}$ .*

Part (i) of the assumption above ensures that there cannot be infinitely many jumps in finite time in the underlying branching process, see, *e.g.*, Athreya and Ney [1, Thm.4.1.1]. Furthermore, under the assertion (iii), there exists a unique solution (up to the boundary) to the stochastic differential equation (SDE) corresponding to  $(\mu, \sigma)$  and so the branching diffusion process is well-defined.

**Assumption 2.3.** *The branching diffusion process goes extinct almost surely.*

It is clearly sufficient to assume that  $\sum_{l \in L} lp_l \leq 1$  for Assumption 2.3 to hold. However, since particles also die when they leave the domain  $\mathcal{O}$ , we can derive weaker conditions. For instance, in the case of branching Brownian motion with exponential lifetime of parameter  $\beta$ , Assumption 2.3 is equivalent to

$$\beta \left( \sum_{l \in L} lp_l - 1 \right) - \frac{\lambda_1}{2} \leq 0, \quad (6)$$

where  $\lambda_1$  is the first positive eigenvalue of the Laplacian in the domain  $\mathcal{O}$ , see Sevast'yanov [26] or Watanabe [30]. See also Remark 2 below for further developments.

### 3 Main Results on Semi-linear PDEs

In this section, we are concerned solely with the semi-linear case, *i.e.*, we assume that  $m = 0$  in Section 2 so that all the particles share the same mark 0 and PDE (5) reads as

$$\begin{aligned} \mathcal{L}u + \beta(f(u) - u) &= 0, & \text{in } \mathcal{O}, \\ u &= h, & \text{on } \partial\mathcal{O}, \end{aligned} \quad (7)$$

where  $f(x, y) = \sum_{l \in L} c_l(x)y^l$ .

**3.1 Probabilistic representation.** We consider a branching diffusion starting from  $x \in \mathcal{O}$  as in Section 2.2. We introduce the following random variable:

$$\psi^x := \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)},$$

where  $\bar{F}(t) := \int_t^{+\infty} \rho(t) dt$  for all  $t \geq 0$  and  $\Delta T_k := T_k - T_{k-}$  is the lifetime of particle  $k$ . For further developments, we also define the underlying diffusion process  $\bar{X}^x$  by

$$\bar{X}_s^x = x + \int_0^s \mu(\bar{X}_r^x) dr + \int_0^s \sigma(\bar{X}_r^x) dW_r, \quad s \geq 0, \mathbb{P} - \text{a.s.},$$

where  $W$  is a  $d$ -dimensional Brownian motion. Denote by  $\eta$  the exit time of the diffusion process  $\bar{X}^x$  from domain  $\mathcal{O}$ , *i.e.*,

$$\eta := \inf \{s \geq 0; \bar{X}_s^x \notin \mathcal{O}\}.$$

**Proposition 1.** *Suppose Assumption 2.1–2.3 hold. If we further assume that PDE (7) has a solution  $u \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$  such that the sequence  $(\psi_n^x)_{n \in \mathbb{N}}$  given by*

$$\psi_n^x := \prod_{\substack{k \in \bigcup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \bigcup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} \prod_{k \in \mathcal{K}^{n+1}} u(X_{T_k}^k),$$

*is uniformly integrable, then we have  $u(x) = \mathbb{E}[\psi^x]$ .*

*Proof.* Following from Itô's formula, we obtain the following Feynman-Kac representation:

$$u(x) = \mathbb{E} \left[ e^{-\beta\eta} h(\bar{X}_\eta^x) + \int_0^\eta \beta e^{-\beta s} f(\cdot, u)(\bar{X}_s^x) ds \right].$$

Next, let  $\tau$  be a positive random variable, independent of  $W$ , with p.d.f. given by  $\rho$ . Then, we can write

$$\begin{aligned} u(x) &= \mathbb{E} \left[ \frac{e^{-\beta\eta} h(\bar{X}_\eta^x)}{\bar{F}(\eta)} \mathbf{1}_{\tau \geq \eta} + \frac{\beta e^{-\beta\tau} f(\cdot, u)(\bar{X}_\tau^x)}{\rho(\tau)} \mathbf{1}_{\tau < \eta} \right] \\ &= \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(X_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} f(\cdot, u)(X_{T_{(1)}}^{(1)})}{\rho(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right] \\ &= \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(X_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} c_{I^{(1)}}(X_{T_{(1)}}^{(1)})}{p_{I^{(1)}} \rho(T_{(1)})} u^{I^{(1)}}(X_{T_{(1)}}^{(1)}) \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right]. \end{aligned} \quad (8)$$

Since an empty product is equal to 1 by convention, it follows that  $u(x) = \mathbb{E}[\psi_1^x]$ . Next, we repeat the above calculations to write for  $k \in \mathcal{K}^2$ ,

$$u(X_{T_k^-}^k) = \mathbb{E} \left[ \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathbf{1}_{X_{T_k}^k \notin \mathcal{O}} + \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} u^{I^k}(X_{T_k}^k) \mathbf{1}_{X_{T_k}^k \in \mathcal{O}} \middle| \mathcal{F}_1 \right]. \quad (9)$$

We use the result in (9) and plug it back in (8) to obtain  $u(x) = \mathbb{E}[\psi_2^x]$ . Similarly, we can show by iteration that for any  $n \geq 1$ , we have  $u(x) = \mathbb{E}[\psi_n^x]$ . To conclude, it remains to observe that  $\psi_n^x$  converges to  $\psi^x$  almost surely in view of Assumption 2.3. Thus, if we suppose that  $(\psi_n^x)_{n \geq 1}$  is uniformly integrable, as  $n \rightarrow \infty$ , we get  $u(x) = \mathbb{E}[\psi^x]$ .  $\square$

Proposition 1 provides a result of uniqueness for a class of semi-linear PDE. Next we establish a result of existence by showing that the stochastic representation is a (viscosity) solution of PDE (7). This is the main result of this section.

**Assumption 3.1.** (i) For any  $\bar{\phi} : \partial\mathcal{O} \rightarrow \mathbb{R}$  continuous, the map  $\bar{\mathcal{O}} \ni x \mapsto \mathbb{E}[e^{-\beta\eta} \bar{\phi}(\bar{X}_\eta^x)]$  is continuous.  
(ii) For any  $\phi : \mathcal{O} \rightarrow \mathbb{R}$  measurable and bounded, the map  $\bar{\mathcal{O}} \ni x \mapsto \mathbb{E}[\phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta}]$  is continuous for  $s > 0$ .

**Theorem 1.** Suppose Assumption 2.1–3.1 hold. If we further assume that  $(\psi^x)_{x \in \mathcal{O}}$  is bounded in  $L^1$ , then  $u : x \mapsto \mathbb{E}[\psi^x]$  is a continuous viscosity solution of PDE (7).

*Proof.* We observe first that, by definition of  $u$ , it holds

$$u(x) = \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(X_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} c_{I^{(1)}}(X_{T_{(1)}}^{(1)})}{p_{I^{(1)}} \rho(T_{(1)})} \tilde{\psi}^x \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right]$$

where

$$\tilde{\psi}^x := \prod_{\substack{k \in \mathcal{K} \setminus \mathcal{K}^1 \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \mathcal{K} \setminus \mathcal{K}^1 \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)}.$$

Further, it follows from Markov property that

$$\mathbb{E} \left[ \tilde{\psi}^x \middle| \mathcal{F}_1 \right] \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} = u^{I^{(1)}}(X_{T_{(1)}}^{(1)}) \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}}.$$

Working backward along the lines of the proof of Proposition 1, we deduce that

$$u(x) = \mathbb{E} \left[ e^{-\beta\eta} h(\bar{X}_\eta^x) + \int_0^\eta \beta e^{-\beta s} f(\cdot, u)(\bar{X}_s^x) ds \right]$$

In view of Assumption 3.1, this last representation yields the continuity of  $u$ . Indeed, since  $h$  is continuous, so is  $x \mapsto \mathbb{E}[e^{-\beta\eta}h(\bar{X}_\eta^x)]$ . In addition,  $u$  being bounded by assumption, Fubini theorem and the dominated convergence theorem yield that the map

$$x \mapsto \mathbb{E} \left[ \int_0^\eta \beta e^{-\beta s} f(\cdot, u)(\bar{X}_s^x) ds \right] = \int_0^{+\infty} \beta e^{-\beta s} \mathbb{E}[f(\cdot, u)(\bar{X}_s^x) \mathbf{1}_{s \leq \eta}] ds$$

is also continuous. Next, if we denote by  $\eta \wedge h$  for any  $h > 0$ , it follows from Markov property that

$$u(x) = \mathbb{E} \left[ e^{-\beta\eta \wedge h} u(\bar{X}_{\eta \wedge h}^x) + \int_0^{\eta \wedge h} \beta e^{-\beta s} f(\cdot, u)(\bar{X}_s^x) ds \right].$$

The fact that  $u$  is a viscosity solution of PDE (7) now follows from classical arguments.  $\square$

**Remark 1.** If we take  $\rho(x) = \beta e^{-\beta x}$ , we recover the probabilistic representation for semi-linear elliptic PDE first derived by Bossy et al. [4], i.e.,

$$u(x) = \mathbb{E} \left[ \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} h(X_{T_k}^k) \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{c_{I^k}(X_{T_k}^k)}{p_{I^k}} \right].$$

The authors provided a result analogous to Proposition 1 and used this representation to estimate the solution of the Poisson-Boltzmann equation.

**3.2 More explicit sufficient conditions.** Let us give more explicit conditions for Theorem 1. We start by a lemma which provides sufficient conditions for Assumption 3.1.

**Lemma 1.** Assume that  $\partial\mathcal{O}$  is of class  $\mathcal{C}^2$  and that  $\sigma$  is uniformly elliptic, i.e., there exists  $\lambda > 0$  such that  $\sigma\sigma^* \geq \lambda I_d$ . Then Assumption 3.1 is satisfied.

*Proof.* (i) Let us study first the continuity of  $\bar{\mathcal{O}} \ni x \mapsto \mathbb{E}[e^{-\beta\eta}\bar{\phi}(\bar{X}_\eta^x)]$  for  $\bar{\phi}$  continuous on  $\partial\mathcal{O}$ . Under the assumptions above, it is well-known that there exists a smooth solution to the following PDE:

$$\begin{aligned} \mathcal{L}u - \beta u &= 0, & \text{in } \mathcal{O}, \\ u &= \bar{\phi}, & \text{on } \partial\mathcal{O}, \end{aligned}$$

see, e.g., Gilbarg and Trudinger [11, Thm.6.13]. By Itô's formula, we deduce that  $u(x) = \mathbb{E}[e^{-\beta\eta}\bar{\phi}(\bar{X}_\eta^x)]$  and the conclusion follows.

(ii) Let us turn now to the continuity of  $\bar{\mathcal{O}} \ni x \mapsto \mathbb{E}[\phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta}]$ . Under the assumption that  $\phi$  is continuous, it is known that the unique smooth solution of

$$\begin{aligned} \partial_t u - \mathcal{L}u &= 0, & \text{in } (0, T] \times \mathcal{O}, \\ u(0, \cdot) &= \phi, & \text{on } \mathcal{O}, \\ u &= 0, & \text{on } (0, T] \times \partial\mathcal{O}, \end{aligned} \tag{10}$$

has the form

$$u(s, x) = \int_{\mathcal{O}} G(s, x; 0, y) \phi(y) dy,$$

where  $G$  is the so-called Green function of PDE (10), see, e.g., Ladyženskaja, Solonnikov and Ural'ceva [21, Thm.4.16.2] or Friedman [10, Thm.3.16]. Then it follows from Itô's formula that

$$\mathbb{E}[\phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta}] = \int_{\mathcal{O}} G(s, x; 0, y) \phi(y) dy.$$

In particular,  $y \mapsto G(s, x; 0, y)$  appears as the density of  $\bar{X}_s^x$  on the event  $\{s \leq \eta\}$ , and so the assumption of continuity on  $\phi$  can be dropped. To conclude, it remains to observe that  $x \mapsto G(s, x; 0, y)$  is continuous such that

$$|G(s, x; 0, y)| \leq Cs^{-\frac{d}{2}} e^{-C\frac{|x-y|^2}{s}},$$

for some constant  $C > 0$ , see [21, Eq.4.16.16].  $\square$

Next we derive explicit conditions to ensure that the family  $(\psi^x)_{x \in \mathcal{O}}$  is uniformly bounded in  $L^1$ . In fact, we go a step further and study boundedness in  $L^2$  to ensure that the corresponding Monte Carlo estimator has finite variance. Let us denote

$$C_1 := \sup_{t \geq 0} \left\{ \frac{e^{-\beta t}}{\bar{F}(t)} \right\} \quad \text{and} \quad C_2 := \sup_{t \geq 0} \left\{ \frac{\beta e^{-\beta t}}{\rho(t)} \right\}.$$

In particular,  $C_1 = C_2 = 1$  when  $\rho(t) = \beta e^{-\beta t}$ . Otherwise  $C_1 \geq 1$  and  $C_2 > 1$ .

**Proposition 2.** *Assume that  $C_1$  and  $C_2$  are finite. If there exists  $v \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$  such that*

$$\begin{aligned} \mathcal{L}v + \beta \left( C_2 \sum_{l \in L} \frac{c_l^2}{p_l} v^l - v \right) &\leq 0, & \text{in } \mathcal{O}, \\ v &\geq C_1 h^2, & \text{on } \partial\mathcal{O}, \end{aligned}$$

then we have  $\mathbb{E}[(\psi^x)^2] \leq v(x)$ . In particular,  $(\psi^x)_{x \in \mathcal{O}}$  is uniformly bounded in  $L^2$ .

*Proof.* First, we observe that Itô's formula yields

$$v(x) \geq \mathbb{E} \left[ C_1 e^{-\beta \eta} h^2(\bar{X}_\eta^x) + C_2 \int_0^\eta \beta e^{-\beta s} \sum_{l \in L} \frac{c_l^2(\bar{X}_s^x)}{p_l} v^l(\bar{X}_s^x) ds \right].$$

Next, by repeating the arguments of Proposition 1, we get

$$v(x) \geq \mathbb{E} \left[ \prod_{\substack{k \in \cup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \notin \mathcal{O}}} \frac{C_1 e^{-\beta \Delta T_k} h^2(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \cup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \in \mathcal{O}}} \frac{C_2 \beta e^{-\beta \Delta T_k} c_{I_k}^2(X_{T_k}^k)}{p_{I_k}^2 \rho(\Delta T_k)} \prod_{k \in \mathcal{K}^{n+1}} v(X_{T_k}^k) \right].$$

Further, it follows from Fatou's lemma that

$$v(x) \geq \mathbb{E} \left[ \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} \frac{C_1 e^{-\beta \Delta T_k} h^2(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{C_2 \beta e^{-\beta \Delta T_k} c_{I_k}^2(X_{T_k}^k)}{p_{I_k}^2 \rho(\Delta T_k)} \right].$$

To conclude, it remains to observe that  $\mathbb{E}[(\psi^x)^2]$  is bounded from above by the right-hand side of the inequality above.  $\square$

We further provide an alternate result towards explicit sufficient condition for uniform boundedness of  $(\psi^x)_{x \in \mathcal{O}}$  in  $L^2$  based on the theory of Markov chains.

$$C_0 := \max \left( C_1 \|h\|_\infty, C_2 \sup_{l \in L} \left\{ \frac{\|c_l\|_\infty}{p_l} \right\} \right).$$

**Lemma 2.** (i) *If  $C_0 \leq 1$ , then  $\psi^x \leq 1$ .*

(ii) *If  $\sum_{l \in L} l p_l < 1$  and  $[-1, 1] \ni s \mapsto \sum_{l \in L} p_l s^l$  is analytic at  $s = 1$ , then there exists  $\gamma > 1$  such that if  $C_0^q < \gamma$  for some  $q \geq 1$ , then  $(\psi^x)_{x \in \mathcal{O}}$  is uniformly bounded in  $L^q$ .*

*Proof.* First, we observe that  $|\psi^x| \leq C_0^{|\mathcal{K}|}$  where  $|\mathcal{K}|$  denotes the cardinality of the set  $\mathcal{K}$ . Thus Part (i) of Lemma 2 is straightforward. For the second assertion, let us consider a branching process with lifetime distribution  $\rho$  and offspring distribution  $(p_l)_{l \in L}$ . Denote by  $N_n$  the number of particles alive at the  $n$ th branching event (arrival time). The process  $(N_n)_{n \in \mathbb{N}}$  is clearly a Markov chain with transition matrix  $P = (P_{i,j})_{i,j \in \mathbb{N}}$  given by

$$P_{0,0} = 1 \quad \text{and} \quad P_{i,i+l-1} = p_l \quad \text{for all } i \geq 1, l \in L.$$

Further it is almost surely absorbed at 0 provided that  $\sum_{l \in L} l p_l \leq 1$ . Next, observe that the extinction time  $\tau$  corresponds to the total number of particles (i.e. the total number of arrivals) in the branching process. Since particles also die when they leave the domain in the branching diffusion process, we can easily construct

a branching process as above coupled with the branching diffusion process such that  $|\mathcal{K}| \leq \tau$ . Further, it follows from Daley [8, Thm.1] that there exists quasi-stationary distribution for the Markov chain  $(N_n)_{n \in \mathbb{N}}$ . In view of Theorem 4 in Coolen-Shrijner and van Doorn [7], this yields the existence of  $\gamma > 1$  such that

$$\mathbb{E}[s^\tau] < +\infty \quad \text{for all } s < \gamma.$$

In addition, it follows from Theorem 3 in [7] that  $\gamma$  can be chosen as follows

$$\gamma := \lim_{n \rightarrow \infty} (\tilde{P}_{i,j}(n))^{-\frac{1}{n}},$$

where  $\tilde{P} = (P_{i,j})_{i,j \geq 1}$  and  $\tilde{P}_{i,j}(n)$  denotes element  $(i,j)$  of the  $n$ th power of  $\tilde{P}$ .  $\square$

**Remark 2.** Consider the following offspring distribution: for  $l \in L \cup \{0\}$ ,

$$\tilde{p}_l := \sup_{x \in \mathcal{O}} \{ \mathbb{P}(|\mathcal{K}^2| = l) \} = p_l + (\mathbf{1}_{l=0} - p_l) \sup_{x \in \mathcal{O}} \{ \mathbb{E}[\bar{F}(\eta)] \}.$$

Clearly, the number of particles in the branching diffusion process is bounded from above by a branching process with lifetime distribution  $\rho$  and offspring distribution  $(\tilde{p}_l)_{l \in L \cup \{0\}}$ . As a consequence, if  $\sum_{l \in L} l \tilde{p}_l \leq 1$ , then the branching diffusion process goes extinct almost surely. In addition, the statement of Lemma 2 still holds true under the weaker assumption  $\sum_{l \in L} l \tilde{p}_l < 1$  and  $[-1, 1] \ni s \mapsto \sum_{l \in L} \tilde{p}_l s^l$  is analytic at  $s = 1$ .

Let us conclude this section by discussing the assumption of uniform integrability of  $(\psi_n^x)_{n \in \mathbb{N}}$  in Proposition 1. As emphasized in Section 5, uniqueness does not hold in general for PDE (7) even if the branching diffusion representation provides a valid solution. As this investigation is out of scope of the present work, we do not pursue it further. However, notice that that under the assumptions of Lemma 2, Proposition 1 yields that  $x \mapsto \mathbb{E}[\psi^x]$  is the unique solution to PDE (7) valued in  $[-1, 1]$  or in  $[-\gamma, \gamma]$  in the subcritical case.

## 4 Main Results on Quasi-linear PDE

In this section we are concerned with the quasi-linear case, *i.e.*, we assume that  $m \geq 1$  in Section 2 so that PDE (5) has a non-linear gradient term and particles in the branching diffusion process carry different marks to account for it. For technical reason, we need to strengthen Assumption 2.1 as follows.

**Assumption 4.1.** The function  $h$  can be extended to a function of class  $\mathcal{C}^{1,\alpha}$  on  $\bar{\mathcal{O}}$ .

**4.1 Probabilistic representation.** Our next assumption is the key automatic differentiation condition on the underlying diffusion process  $\bar{X}^x$ . Recall that

$$\bar{X}_s^x = x + \int_0^s \mu(\bar{X}_r^x) dr + \int_0^s \sigma(\bar{X}_r^x) dW_r, \quad s \geq 0, \mathbb{P} - \text{a.s.},$$

and  $\eta$  denotes the exit time of  $\bar{X}^x$  from the domain  $\mathcal{O}$ .

**Assumption 4.2.** (i) For any  $\bar{\phi} \in \mathcal{C}^{1,\alpha}(\bar{\mathcal{O}})$ , the map  $\mathcal{O} \ni x \mapsto \mathbb{E}[e^{-\beta\eta} \bar{\phi}(\bar{X}_\eta^x)]$  is continuously differentiable and there exists a measurable function  $\mathcal{W}_{\partial\mathcal{O}}(x, (W_r)_{r \in [0, \eta]})$  such that

$$\partial_x \mathbb{E}[e^{-\beta\eta} \bar{\phi}(\bar{X}_\eta^x)] = \mathbb{E}[e^{-\beta\eta} \bar{\phi}(\bar{X}_\eta^x) \mathcal{W}_{\partial\mathcal{O}}(x, (W_r)_{r \in [0, \eta]})].$$

(ii) For any  $\phi : \mathcal{O} \rightarrow \mathbb{R}$  measurable and bounded, the map  $\mathcal{O} \ni x \mapsto \mathbb{E}[\int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) ds]$  is continuously differentiable and there exists a measurable function  $\mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]})$  such that

$$\partial_x \mathbb{E}\left[\int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) ds\right] = \mathbb{E}\left[\int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) \mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]}) ds\right].$$

Let us define

$$\bar{\mathcal{W}}(s, x, (W_r)_{r \in [0, s]}) := \mathcal{W}_{\partial\mathcal{O}}(x, (W_r)_{r \in [0, \eta]}) \mathbf{1}_{\bar{X}_s^x \notin \mathcal{O}} + \mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]}) \mathbf{1}_{\bar{X}_s^x \in \mathcal{O}}$$

We consider a marked branching diffusion process starting from  $x \in \mathcal{O}$  as explained in Section 2.2 and denote

$$\mathcal{W}_k = \mathbf{1}_{m_k=0} + \mathbf{1}_{m_k \neq 0} b_{m_k}(X_{T_k-}^k) \cdot \bar{\mathcal{W}}(\Delta T_k, X_{T_k-}^k, \Delta W^k),$$



where  $m_k$  denotes the mark of particle  $k$ . We next introduce the following random variable:

$$\psi^x := \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathcal{W}_k \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} \mathcal{W}_k.$$

**Proposition 3.** *Suppose Assumption 2.1–2.3 and 4.1–4.2 hold and further assume that PDE (5) has a solution  $u \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}^1(\bar{\mathcal{O}})$  such that the sequence  $(\psi_n^x)_{n \in \mathbb{N}}$  defined by*

$$\psi_n^x := \prod_{\substack{k \in \bigcup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathcal{W}_k \prod_{\substack{k \in \bigcup_{i=1}^n \mathcal{K}^i \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} \mathcal{W}_k \prod_{\substack{k \in \mathcal{K}^{n+1} \\ m_k=0}} u(X_{T_k}^k) \prod_{i=1}^m \prod_{\substack{k \in \mathcal{K}^{n+1} \\ m_k=i}} (b_i \cdot Du)(X_{T_k}^k),$$

is uniformly integrable, then we have  $u(x) = \mathbb{E}[\psi^x]$ .

*Proof.* Following from Itô's formula, we have the Feynman-Kac representation:

$$u(x) = \mathbb{E} \left[ e^{-\beta \eta} h(\bar{X}_\eta^x) + \int_0^\eta \beta e^{-\beta s} f(\cdot, u, Du)(\bar{X}_s^x) ds \right]$$

Next, let  $\tau$  be a positive random variable, independent of  $W$ , with p.d.f. given by  $\rho$ . Then, we can write

$$\begin{aligned} u(x) &= \mathbb{E} \left[ \frac{e^{-\beta \eta} h(\bar{X}_\eta^x)}{\bar{F}(\eta)} \mathbf{1}_{\tau \geq \eta} + \frac{\beta e^{-\beta \tau} f(\cdot, u, Du)(\bar{X}_\tau^x)}{\rho(\tau)} \mathbf{1}_{\tau < \eta} \right] \\ &= \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(\bar{X}_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} f(\cdot, u, Du)(X_{T_{(1)}}^{(1)})}{\rho(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right] \\ &= \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(\bar{X}_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} c_{I^{(1)}}(X_{T_{(1)}}^{(1)})}{p_{I^{(1)}} \rho(T_{(1)})} u^{I_0^{(1)}}(X_{T_{(1)}}^{(1)}) \prod_{i=1}^m (b_i \cdot Du)^{I_i^{(1)}}(X_{T_{(1)}}^{(1)}) \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right]. \end{aligned} \quad (11)$$

Since an empty product is equal to 1 by convention, it follows that  $u(x) = \mathbb{E}[\psi_1^x]$ . Next, we repeat the above calculations to write for  $k \in \mathcal{K}^2$ ,

$$u(X_{T_k}^k) = \mathbb{E} \left[ \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathbf{1}_{X_{T_k}^k \notin \mathcal{O}} + \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} u^{I_0^k}(X_{T_k}^k) \prod_{i=1}^m (b_i \cdot Du)^{I_i^k}(X_{T_k}^k) \mathbf{1}_{X_{T_k}^k \in \mathcal{O}} \middle| \mathcal{F}_1 \right] \quad (12)$$

Further, in the original Feynman-Kac formula, we obtain by differentiating and using Assumption 4.2,

$$\begin{aligned} Du(x) &= \mathbb{E} \left[ e^{-\beta \eta} h(\bar{X}_\eta^x) \mathcal{W}_{\partial \mathcal{O}}(x, (W_r)_{r \in [0, \eta]}) + \int_0^\eta \beta e^{-\beta s} f(\cdot, u, Du)(\bar{X}_s^x) \mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]}) ds \right] \\ &= \mathbb{E} \left[ \psi_1^x \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}) \right]. \end{aligned}$$

Similarly, for  $k \in \mathcal{K}^2$ ,

$$\begin{aligned} Du(X_{T_k}^k) &= \mathbb{E} \left[ \left( \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathbf{1}_{X_{T_k}^k \notin \mathcal{O}} + \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} u^{I_0^k}(X_{T_k}^k) \prod_{i=1}^m (b_i \cdot Du)^{I_i^k}(X_{T_k}^k) \mathbf{1}_{X_{T_k}^k \in \mathcal{O}} \right) \right. \\ &\quad \left. \bar{\mathcal{W}}(\Delta T_k, X_{T_k}^k, \Delta W^k) \middle| \mathcal{F}_1 \right]. \quad (13) \end{aligned}$$

We use the results in (12) and (13) and plug them back in (11) to obtain  $u(x) = \mathbb{E}[\psi_2^x]$ . Similarly, we can show by iteration that for any  $n \geq 1$ , we have  $u(x) = \mathbb{E}[\psi_n^x]$ . To conclude, it remains to observe that, under Assumption 2.3,  $\psi_n^x$  converges to  $\psi^x$  almost surely. Thus, if we suppose  $(\psi_n^x)_{n \geq 1}$  is uniformly integrable, as  $n \rightarrow \infty$ , we get  $u(x) = \mathbb{E}[\psi^x]$ .  $\square$

Proposition 3 provides a result of uniqueness for a class of quasi-linear PDE. Next we establish a result of existence by showing that the stochastic representation is a viscosity solution of PDE (5).

**Theorem 2.** *Suppose Assumption 2.1–2.3 and 4.1–4.2 hold. If we further assume that  $(\psi^x)_{x \in \mathcal{O}}$  and, for all  $i = 1, \dots, m$ ,  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  are uniformly bounded in  $L^1$ , then  $u : x \mapsto \mathbb{E}[\psi^x]$  is a continuously differentiable viscosity solution of PDE (5).*

*Proof.* We observe first that by definition of  $u$ , it holds

$$u(x) = \mathbb{E} \left[ \frac{e^{-\beta T_{(1)}} h(X_{T_{(1)}}^{(1)})}{\bar{F}(T_{(1)})} \mathbf{1}_{X_{T_{(1)}}^{(1)} \notin \mathcal{O}} + \frac{\beta e^{-\beta T_{(1)}} c_{I^{(1)}}(X_{T_{(1)}}^{(1)})}{p_{I^{(1)}} \rho(T_{(1)})} \tilde{\psi}^x \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} \right]$$

where

$$\tilde{\psi}^x := \prod_{\substack{k \in \mathcal{K} \setminus \mathcal{K}^1 \\ X_{T_k}^k \notin \mathcal{O}}} \frac{e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \mathcal{W}_k \prod_{\substack{k \in \mathcal{K} \setminus \mathcal{K}^1 \\ X_{T_k}^k \in \mathcal{O}}} \frac{\beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)} \mathcal{W}_k.$$

Further, it follows from Markov property that

$$\mathbb{E} \left[ \tilde{\psi}^x \middle| \mathcal{F}_1 \right] \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}} = u^{I_0^{(1)}} \left( X_{T_{(1)}}^{(1)} \right) \prod_{i=1}^m v_i^{I_i^{(1)}} \left( X_{T_{(1)}}^{(1)} \right) \mathbf{1}_{X_{T_{(1)}}^{(1)} \in \mathcal{O}}.$$

where  $(v_i)_{i=1, \dots, m}$ ,  $v_i : \mathcal{O} \mapsto \mathbb{R}$  are given by

$$v_i(x) := \mathbb{E}[\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)})].$$

Working backward along the lines of the proof of Proposition 3, we deduce that

$$u(x) = \mathbb{E} \left[ e^{-\beta \eta} h(\bar{X}_\eta^x) + \int_0^\eta \beta e^{-\beta s} \left( \sum_{l \in L} c_l u^{l_0} \prod_{i=1}^m v_i^{l_i} \right) (\bar{X}_s^x) ds \right]$$

In particular, since  $u$  and  $(v_i)_{i=1, \dots, m}$  are bounded by assumption, it follows from Assumption 4.2 that  $u$  is continuously differentiable and

$$\begin{aligned} Du(x) &= \mathbb{E} \left[ e^{-\beta \eta} h(\bar{X}_\eta^x) \mathcal{W}_{\partial \mathcal{O}}(x, (W_r)_{r \in [0, \eta]}) + \int_0^\eta \beta e^{-\beta s} \left( \sum_{l \in L} c_l u^{l_0} \prod_{i=1}^m v_i^{l_i} \right) (\bar{X}_s^x) \mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]}) ds \right] \\ &= \mathbb{E}[\psi^x \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)})]. \end{aligned}$$

Thus, for all  $i = 1, \dots, m$ ,  $v_i$  coincides with  $b_i \cdot Du$  and it holds

$$u(x) = \mathbb{E} \left[ e^{-\beta \eta} h(\bar{X}_\eta^x) + \int_0^\eta \beta e^{-\beta s} f(\cdot, u, Du)(\bar{X}_s^x) ds \right].$$

Next, it follows from Markov property that for any  $h > 0$ ,

$$u(x) = \mathbb{E} \left[ e^{-\beta \eta \wedge h} u(\bar{X}_{\eta \wedge h}^x) + \int_0^{\eta \wedge h} \beta e^{-\beta s} f(\cdot, u, Du)(\bar{X}_s^x) ds \right].$$

The fact that  $u$  is a viscosity solution of PDE (5) now follows by classical arguments.  $\square$

**4.2 Automatic differentiation formula: the general case** The aim of this section is to provide sufficient conditions to ensure that Assumption 4.2 holds and to derive explicit formula for  $\bar{\mathcal{W}}$ . This automatic differentiation formula originates from Thalmaier [28, 29] and was subsequently developed by Delarue [9] and Gobet [12].

**Assumption 4.3.** (i) *The coefficients  $(\mu, \sigma)$  are continuously differentiable on  $\bar{\mathcal{O}}$ . In addition, their partial derivatives are Hölder continuous.*

(ii) *The diffusion coefficient  $\sigma$  is uniformly elliptic, i.e., there exists  $\lambda > 0$  such that  $\sigma \sigma^* \geq \lambda I_d$ .*

(iii) *The boundary  $\partial \mathcal{O}$  is of class  $\mathcal{C}^2$ .*

We start by establishing a technical lemma. Fix a finite horizon  $T > 0$  and denote for all  $s \geq 0$ ,

$$\theta_s(r, y) := \frac{1}{d(y, \partial \mathcal{O})^2 (s \wedge T - r)}.$$

**Lemma 3.** *Under Assumption 4.3 (iii), it holds for all  $x \in \mathcal{O}$  and  $s \geq 0$ ,*

$$\int_0^{\eta \wedge s \wedge T} \theta_s(r, \bar{X}_r^x) dr = +\infty. \quad (14)$$

*In addition, if we denote*

$$\zeta_s := \inf \left\{ t > 0 : \int_0^t \theta_s(r, \bar{X}_r^x) dr = 1 \right\}, \quad (15)$$

*then there exists  $s' < s \wedge T$  such that  $\zeta_s \leq \eta \wedge s'$  and for all  $q \geq 1$ ,*

$$\mathbb{E} \left[ \left( \int_0^{\zeta_s} \theta_s^2(r, \bar{X}_r^x) dr \right)^q \right] \leq \frac{C}{d(x, \partial\mathcal{O})^{4q-2} (s \wedge T)^q}$$

*where  $C > 0$  depends on  $T$  but not on  $x$  or  $s$ .*

*Proof.* The proof essentially follows from Delarue [9]. More precisely, let us assume first that  $\mathcal{O}$  is a ball. Then Proposition 2.3 in [9] ensures that

$$\int_0^{\eta \wedge s \wedge T} l(r, \bar{X}_r^x)^{-2} dr = +\infty,$$

where  $l(r, y) := d(y, \partial\mathcal{O})(s \wedge T - r)$ . In addition, Proposition 2.4 in [9] yields that

$$\mathbb{E} \left[ \left( \int_0^{\zeta_s} l(r, \bar{X}_r^x)^{-4} dr \right)^q \right] \leq \frac{C}{d(x, \partial\mathcal{O})^{4q-2} (s \wedge T)^{3q}}.$$

Both identities (14) and (15) are easily obtained by repeating the arguments in [9] with the map  $(r, y) \mapsto d(y, \partial\mathcal{O})\sqrt{s \wedge T - r}$  instead of  $l$ . In particular, it follows from (14) that  $\zeta_s \leq \eta \wedge s \wedge T$ . Further, it holds for all  $s' \leq s \wedge T$ ,

$$\int_0^{s'} \theta_s(r, \bar{X}_r^x) dr \mathbf{1}_{s' \leq \eta} \geq -C^{-1} \log \left( 1 - \frac{s'}{s \wedge T} \right) \mathbf{1}_{s' \leq \eta}$$

where  $C := \text{diam}(\mathcal{O})^2/4$ . Thus for  $s' = (1 - e^{-C})(s \wedge T)$ , we have  $\zeta_s \leq \eta \wedge s'$ . For the general case, it remains to observe that we can repeat the arguments in [9] by using a  $\mathcal{C}^2$ -extension of the distance to the boundary as in Proposition 2.1 of Gobet and Menozzi [13].  $\square$

**Proposition 4.** *Under Assumption 4.3, the assertions of Assumption 4.2 are satisfied with*

$$\begin{aligned} \mathcal{W}_{\partial\mathcal{O}}(x, (W_r)_{r \in [0, \eta]}) &= \int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r, \\ \mathcal{W}_{\mathcal{O}}(s, x, (W_r)_{r \in [0, s]}) &= \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r, \end{aligned}$$

*where  $Y^x$  is the tangent process given by*

$$Y_s^x = I_d + \int_0^s Db(\bar{X}_r^x) Y_r^x dr + \int_0^s \sum_{i=1}^d D\sigma_i(\bar{X}_r^x) Y_r^x dW_r^i.$$

*In addition, it holds for all  $q \geq 1$ ,*

$$\mathbb{E} \left[ \left| \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r \right|^q \right] \leq \frac{C}{d(x, \partial\mathcal{O})^{2q-1} s^{\frac{q}{2}}}.$$

This result is a slight extension of the automatic differentiation formula obtained by Delarue [9] and Gobet [12]. We postpone the proof to the appendix. Note that the estimate is an easy consequence of

**Lemma 3.** Indeed, using successively the Burkholder-Davis-Gundy inequality, the boundedness of  $\sigma^{-1}$  and the fact that  $\sup_{0 \leq r \leq s} \{|Y_r^x|\}$  has finite moments, we derive for all  $s \leq T$ ,

$$\begin{aligned} \mathbb{E} \left[ \left| \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r \right|^q \right] &\leq C \mathbb{E} \left[ \sup_{0 \leq r \leq s} \{|Y_r^x|^q\} \left( \int_0^{\zeta_s} \theta_s^2(r, \bar{X}_r^x) dr \right)^{\frac{q}{2}} \right] \\ &\leq C \mathbb{E} \left[ \left( \int_0^{\zeta_s} \theta_s^2(r, \bar{X}_r^x) dr \right)^q \right]^{1/2} \\ &\leq \frac{C}{d(x, \partial \mathcal{O})^{2q-1} s^{\frac{q}{2}}}. \end{aligned}$$

**Remark 3.** The choice of automatic differentiation weight function is not unique. Actually we can replace  $(\theta_s(r, \bar{X}_r^x))_{r \geq 0}$  by (almost) any predictable process  $(h_s(r))_{r \geq 0}$  such that

$$h_s(r) = 0 \quad \text{for all } r \geq \eta \wedge s \quad \text{and} \quad \int_0^{\eta \wedge s} h_s(r) dr = 1.$$

We now provide sufficient conditions to ensure that  $(\psi^x)_{x \in \mathcal{O}}$  and  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  are uniformly bounded in  $L^q$  for  $q \geq 1$ . Let us define

$$\begin{aligned} C_{1,q} &:= \sup_{t \geq 0} \left\{ \frac{e^{-\beta t}}{\bar{F}(t)} \right\} \sup_{x \in \partial \mathcal{O}} \{|h(x)|\} \sup_{x \in \mathcal{O}, i=1, \dots, m} \left\{ \mathbb{E} \left[ |b_i(x) \cdot \mathcal{W}_{\partial \mathcal{O}}(x, (W_r)_{r \in [0, \eta]})|^q \right]^{\frac{1}{q}} \right\}, \\ C_{2,q} &:= \sup_{t \geq 0, x \in \mathcal{O}, l \in L, i=1, \dots, m} \left\{ \frac{\beta e^{-\beta t} |c_l(x)|}{p_l \rho(t)} \mathbb{E} \left[ |b_i(x) \cdot \mathcal{W}_{\mathcal{O}}(t, x, (W_r)_{r \in [0, t]})|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Using the estimates from Proposition 4,  $C_{1,q}$  and  $C_{2,q}$  are clearly finite if the following condition is satisfied:

$$\sup_{t \geq 0} \left\{ \frac{e^{-\beta t}}{\bar{F}(t)} \right\} \vee \sup_{t \geq 0} \left\{ \frac{\beta e^{-\beta t}}{\sqrt{t} \rho(t)} \right\} \vee \sup_{l \in L} \left\{ \frac{\|c_l\|_{\infty}}{p_l} \right\} \vee \sup_{x \in \mathcal{O}, i=1, \dots, m} \left\{ \frac{|b_i(x)|}{d(x, \partial \mathcal{O})^{2-\frac{1}{q}}} \right\} < +\infty.$$

In order to ensure that the first two elements above are finite, one needs to choose a convenient lifetime distribution. For instance, we can take a Gamma distribution with shape parameter 0.5 and rate parameter  $\beta' < \beta$ , i.e.,  $\rho(s) = \sqrt{\beta' / (\pi s)} e^{-\beta' s}$ . Another choice consists in taking for any  $\beta' > 0$ ,  $\rho(s) = \beta' / (2\sqrt{s}) e^{-\beta' \sqrt{s}}$ .

**Lemma 4.** Denote  $C_{0,q} := C_{1,q} \vee C_{2,q}$ .

- (i) If  $C_{0,q} \leq 1$ , then  $(\psi^x)_{x \in \mathcal{O}}$  and  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  are bounded by 1 in  $L^q$ .
- (ii) If  $\sum_{l \in L} l p_l < 1$ , then there exists  $\gamma > 1$  such that if  $C_{0,q}^q < \gamma$ , then  $(\psi^x)_{x \in \mathcal{O}}$  and  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  are uniformly bounded in  $L^q$ .

*Proof.* Let us denote by  $\mathcal{F}_0$  the  $\sigma$ -algebra generated by the underlying branching process, i.e.,

$$\mathcal{F}_0 := \sigma \left( \tau^{n, \tilde{n}}, I^{n, \tilde{n}}, n \geq 1, \tilde{n} \geq 1 \right).$$

Conditioning by  $\mathcal{F}_0$ , we obtain for all  $x \in \mathcal{O}$ ,

$$\mathbb{E} [|\psi^x|^q] \leq \mathbb{E} \left[ C_{0,q}^{q|\mathcal{K}|} \right]$$

and the same holds true if we consider  $\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)})$  instead of  $\psi^x$ . Thus the desired result follows by the same arguments as in Lemma 2.  $\square$

**4.3 Automatic differentiation formula: the one-dimensional case** The automatic differentiation formula given by Proposition 4, though quite general, might not be very efficient for numerical applications. Indeed, one needs to compute a stochastic integral for each particle that holds a non-zero mark. In this section, we provide a simpler automatic differentiation formula that is satisfied for a one-dimensional Brownian motion exiting from an interval.

Let us assume that  $d = 1$ ,  $b = 0$ ,  $\sigma = 1$  and  $\mathcal{O} = [-r, r]$  for some  $r > 0$ . Both lemmas below show that Assumption 4.3 is satisfied in this setting.

**Lemma 5.** For any  $\phi : (-r, r) \rightarrow \mathbb{R}$  measurable and bounded, the map  $(-r, r) \ni x \mapsto \mathbb{E}[\int_0^\eta e^{-\beta s} \phi(W_s^x) ds]$  is continuously differentiable and

$$\partial_x \mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(W_s^x) ds \right] = \mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(W_s^x) \mathcal{W}_\mathcal{O}(x, W_s^x) ds \right],$$

where  $\mathcal{W}_\mathcal{O} : (-r, r)^2 \rightarrow \mathbb{R}$  is given by

$$\mathcal{W}_\mathcal{O}(x, y) := \begin{cases} \sqrt{2\beta} \frac{\sqrt{2\beta}}{\tanh(\sqrt{2\beta}(r+x))}, & \text{if } y > x, \\ -\frac{\sqrt{2\beta}}{\tanh(\sqrt{2\beta}(r-x))}, & \text{if } y \leq x. \end{cases}$$

*Proof.* Let us denote

$$u(x) = \mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(W_s^x) ds \right].$$

We first assume that  $\phi$  is continuous. Then  $u$  satisfies the following ODE:

$$\frac{1}{2}u'' - \beta u = -\phi, \quad \text{in } (-r, r), \quad (16)$$

with the boundary conditions  $u(-r) = u(r) = 0$ . By standard arguments, including variation of parameters, we can solve this ODE explicitly. We obtain

$$u(x) = \int_{-r}^r G(x, y) \phi(y) dy, \quad (17)$$

where  $G : (-r, r)^2 \rightarrow \mathbb{R}$  is given by

$$G(x, y) := \frac{-2}{\sqrt{2\beta}} \left( \sinh(\sqrt{2\beta}(x-y)_+) - \frac{\sinh(\sqrt{2\beta}(r+x))}{\sinh(2\sqrt{2\beta}r)} \sinh(\sqrt{2\beta}(r-y)) \right).$$

By a standard approximation procedure, it is clear that (17) still holds true if  $\phi$  is only assumed measurable and bounded. Further, a direct calculation yields that

$$\frac{\partial_x G(x, y)}{G(x, y)} = \begin{cases} \frac{\sqrt{2\beta}}{\tanh(\sqrt{2\beta}(r+x))}, & \text{if } y > x, \\ -\frac{\sqrt{2\beta}}{\tanh(\sqrt{2\beta}(r-x))}, & \text{if } y \leq x. \end{cases}$$

The desired result then follows by differentiation under the integral sign in (17).  $\square$

**Lemma 6.** For any  $\bar{\phi} : \{-r, r\} \rightarrow \mathbb{R}$ , the map  $(-r, r) \ni x \mapsto \mathbb{E}[e^{-\beta\eta} \bar{\phi}(W_\eta^x)]$  is continuously differentiable and

$$\partial_x \mathbb{E} [e^{-\beta\eta} \bar{\phi}(W_\eta^x)] = \mathbb{E} [e^{-\beta\eta} \bar{\phi}(W_\eta^x) \mathcal{W}_{\partial\mathcal{O}}(W_\eta^x)],$$

where  $\mathcal{W}_{\partial\mathcal{O}} : \{-r, r\} \rightarrow \mathbb{R}$  is given by

$$\mathcal{W}_{\partial\mathcal{O}}(y) := \begin{cases} \sqrt{2\beta} \left( \frac{\bar{\phi}(r)}{\tanh(2\sqrt{2\beta}r)} - \frac{\bar{\phi}(-r)}{\sinh(2\sqrt{2\beta}r)} \right), & \text{if } y = r, \\ \sqrt{2\beta} \left( \frac{\bar{\phi}(-r)}{\tanh(-2\sqrt{2\beta}r)} - \frac{\bar{\phi}(r)}{\sinh(-2\sqrt{2\beta}r)} \right), & \text{if } y = -r. \end{cases}$$

*Proof.* Let us denote

$$u(x) = \mathbb{E} [e^{-\beta\eta} \bar{\phi}(W_\eta^x)].$$

It satisfies the following ODE:

$$\frac{1}{2}u'' - \beta u = 0, \quad \text{in } (-r, r), \quad (18)$$

with the boundary conditions  $u(-r) = \bar{\phi}(-r)$  and  $u(r) = \bar{\phi}(r)$ . It follows that

$$u(x) = \frac{\bar{\phi}(r) \sinh(\sqrt{2\beta}(r+x)) + \bar{\phi}(-r) \sinh(\sqrt{2\beta}(r-x))}{\sinh(2\sqrt{2\beta}r)}$$

To conclude, it remains to observe that  $u'$  is a solution to ODE (18) satisfying the boundary conditions

$$\begin{aligned} u'(r) &= \sqrt{2\beta} \left( \frac{\bar{\phi}(r)}{\tanh(2\sqrt{2\beta}r)} - \frac{\bar{\phi}(-r)}{\sinh(2\sqrt{2\beta}r)} \right), \\ u'(-r) &= \sqrt{2\beta} \left( \frac{\bar{\phi}(-r)}{\tanh(-2\sqrt{2\beta}r)} - \frac{\bar{\phi}(r)}{\sinh(-2\sqrt{2\beta}r)} \right). \end{aligned}$$

□

**Remark 4.** It is remarkable that the weight  $\mathcal{W}_{\partial\mathcal{O}}$  does not depend on the starting point  $x$  in this setting. This is due to the fact that both  $u$  and  $u'$  in the proof of Lemma 6 satisfy ODE (18) and thus one only needs to account for their values at the boundary of the domain. On the contrary,  $\mathcal{W}_{\mathcal{O}}$  explodes when  $x$  approaches the boundary as in Section 4.2. This is expected, since when  $u$  is a solution to ODE (16) vanishing at the boundary, its derivative  $u'$  does not vanish at the boundary in general.

We provide now sufficient conditions to ensure that  $(\psi^x)_{x \in \mathcal{O}}$  and  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  are uniformly bounded in  $L^q$ . Let us define

$$\begin{aligned} C_1 &:= \sup_{x \in \mathcal{O}, i=1, \dots, m} \{|b_i(x)|\} \max \{|\mathcal{W}_{\partial\mathcal{O}}(l)|, |\mathcal{W}_{\partial\mathcal{O}}(-l)|\}, \\ C_2 &:= \sup_{x, y \in \mathcal{O}, i=1, \dots, m} \{|b_i(x) \cdot \mathcal{W}_{\mathcal{O}}(x, y)|\}. \end{aligned}$$

Clearly  $C_2$  is finite if the following condition is satisfied:

$$\sup_{x \in \mathcal{O}, i=1, \dots, m} \left\{ \frac{|b_i(x)|}{d(x, \partial\mathcal{O})} \right\} < +\infty.$$

Now we consider the family  $(\bar{\psi}^x)_{x \in \mathcal{O}}$  given by

$$\bar{\psi}^x := \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} \frac{(C_1 \vee 1) e^{-\beta \Delta T_k} h(X_{T_k}^k)}{\bar{F}(\Delta T_k)} \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{(C_2 \vee 1) \beta e^{-\beta \Delta T_k} c_{I^k}(X_{T_k}^k)}{p_{I^k} \rho(\Delta T_k)}.$$

The tools developed in Section 3.2 allow to derive sufficient conditions to ensure that the family  $(\bar{\psi}^x)_{x \in \mathcal{O}}$  is bounded in  $L^q$ . Since  $|\psi^x| \leq |\bar{\psi}^x|$ , the boundedness on  $(\psi^x)_{x \in \mathcal{O}}$  in  $L^q$  follows immediately. The same holds true for  $(\psi^x b_i(x) \cdot \bar{\mathcal{W}}(T_{(1)}, x, \Delta W^{(1)}))_{x \in \mathcal{O}}$  for all  $i = 1, \dots, m$ .

## 5 Numerical Examples with Related Technical Discussions

**5.1 Example 1.** We first start with the following semi-linear PDE

$$u'' = \frac{2(k+1)}{(k-1)^2} u^k, \quad k > 1, \tag{19}$$

which has an explicit solution given as

$$u(x) = \frac{1}{(x + x_0)^{\frac{2}{k-1}}},$$

where  $x_0$  is a constant. To test the validity of our probabilistic representation in Proposition 1, we consider for  $k = 2$  in (19)

$$\begin{aligned} \frac{1}{2} u'' + \beta \left( -\frac{3}{\beta} u^2 + u - u \right) &= 0, \quad \text{in } \mathcal{O} = \left( -\frac{1}{2}, \frac{1}{2} \right), \\ u\left(-\frac{1}{2}\right) &= \frac{4}{9}, \quad u\left(\frac{1}{2}\right) = \frac{4}{25}. \end{aligned}$$

The solution of the above PDE is given as  $u(x) = \frac{1}{(x+2)^2}$ . From our previous results, it is clear that we can compute  $u(x)$  as

$$u(x) = \mathbb{E} \left[ \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \notin \mathcal{O}}} h(X_{T_k}^k) \prod_{\substack{k \in \mathcal{K} \\ X_{T_k}^k \in \mathcal{O}}} \frac{c_{I^k}(X_{T_k}^k)}{p_{I^k}} \right], \quad (20)$$

with branching times distributed exponentially with intensity  $\beta > 0$ ,  $L = \{1, 2\}$ ,  $c_1(x) = 1$  and  $c_2(x) = -3/\beta$ . In this example, we can implement an unbiased estimator as the underlying diffusion process corresponds to a Brownian motion. To achieve this, we generate samples for exit time and position of a Brownian motion from an interval with finite time horizon using the work by Lejay [22]. In Table 5.1, we illustrate numerical results for different values of starting position  $x$  and probability mass function (p.m.f.)  $\{p_l\}_{l \in L}$  with  $1 \times 10^6$  Monte Carlo sample paths and  $\beta = 1.0$ .

$x$	$\{p_1, p_2\}$	Estimate	99% conf. interval	Std. Dev./Mean	Relative error	Run time (secs)
0	{0.5, 0.5}	0.247	[0.242, 0.251]	6.415	1.36%	21
0	{0.2, 0.8}	0.246	[0.242, 0.250]	6.969	1.55%	23
0	{0.8, 0.2}	0.250	[0.238, 0.259]	16.526	0.46%	19
-0.2	{0.5, 0.5}	0.308	[0.303, 0.312]	6.415	0.33%	27
-0.2	{0.2, 0.8}	0.310	[0.302, 0.317]	9.044	0.33%	29
-0.2	{0.8, 0.2}	0.306	[0.298, 0.313]	9.286	0.83%	25

Table 5.1: Numerical results for the unbiased estimator of probabilistic representation (20)

From Table 5.1, it is clear that the choice of  $\{p_l\}_{l \in L} = \{0.5, 0.5\}$  provides the most accurate estimation results. We thus compare its performance with an intuitive choice of p.m.f. given by  $p_1^* = \beta/(\beta + 3)$  and  $p_2^* = 3/(\beta + 3)$  for different values of  $\beta$  and  $x = 0.2$  in Table 5.2.

$\beta$	$\{p_1^*, p_2^*\}$			$\{0.5, 0.5\}$		
	99% conf. interval	Std. Dev./Mean	Run time (secs)	99% conf. interval	Std. Dev./Mean	Run time (secs)
5.0	[0.204, 0.210]	6.48	91	[0.199, 0.233]	30.22	114
4.0	[0.205, 0.211]	6.09	75	[0.198, 0.210]	11.74	114
3.0	[0.204, 0.209]	5.19	60	[0.203, 0.208]	4.81	58
2.5	[0.204, 0.210]	5.23	50	[0.204, 0.209]	4.64	48
2.0	[0.201, 0.208]	7.17	42	[0.204, 0.209]	4.60	40
1.5	[0.204, 0.213]	7.80	35	[0.204, 0.213]	8.22	33
1.0	[0.199, 0.213]	12.71	28	[0.204, 0.211]	7.389	27

Table 5.2: Comparison of performance for different values of  $\beta$  between two choices of p.m.f.

In Table 5.2, we observe that the estimator corresponding to the choice of p.m.f.  $= \{p_1^*, p_2^*\}$  is robust with respect to different values of  $\beta$ . The numerical results exhibit that our method is accurate in estimating the true solution of semi-linear PDE of the type (7) in one dimension.

**5.2 Example 2.** We consider another semi-linear PDE in one dimension

$$u'' - u + u^3 = 0, \quad (21)$$

with explicit solution  $u(x) = \frac{\sqrt{2}}{\cosh(x)}$ . Once again, to test the validity of our probabilistic representation, we reformulate the above PDE as

$$\frac{1}{2}u'' + \beta \left( \frac{1}{2\beta}u^3 + \frac{\beta - 0.5}{\beta}u - u \right) = 0, \quad \text{in } \mathcal{O} = \left( -\frac{1}{2}, \frac{1}{2} \right),$$

$$u(-0.5) = \frac{\sqrt{2}}{\cosh(-0.5)}, \quad u(0.5) = \frac{\sqrt{2}}{\cosh(0.5)}.$$

In Table 5.3, we illustrate numerical results for different values of parameters with  $1 \times 10^6$  Monte Carlo sample paths and  $\beta = 1.0$  which exhibit that our estimator is accurate.

$x$	$[p_1, p_2]$	Estimate	99% conf. interval	Std. Dev./Mean	Relative error	Run time (secs)
0	[0.5, 0.5]	1.414	[1.410, 1.418]	1.049	0.02%	25
0	[0.2, 0.8]	1.415	[1.412, 1.418]	0.780	0.06%	31
0	[0.8, 0.2]	1.401	[1.389, 1.413]	3.362	0.94%	22
-0.2	[0.5, 0.5]	1.384	[1.382, 1.387]	0.779	0.14%	30
-0.2	[0.2, 0.8]	1.387	[1.382, 1.392]	1.398	0.02%	36
-0.2	[0.8, 0.2]	1.378	[1.363, 1.392]	4.067	0.61%	27

Table 5.3: Numerical results for the unbiased estimator of probabilistic representation of semi-linear PDE (21)

In Table 5.4, we once more compare the performance of p.m.f. choice  $\{0.5, 0.5\}$  with the intuitive choice  $p_1^* = 2|\beta - 0.5|/(2|\beta - 0.5| + 1)$  and  $p_3^* = 1/(2|\beta - 0.5| + 1)$  for different values of  $\beta$  and  $x = 0.2$ .

$\beta$	$\{p_1^*, p_2^*\}$			$\{0.5, 0.5\}$		
	99% conf. interval	Std. Dev./Mean	Run time (secs)	99% conf. interval	Std. Dev./Mean	Run time (secs)
4.0	[1.383, 1.392]	1.152	61	[1.382, 1.394]	1.603	274
3.0	[1.382, 1.390]	1.140	51	[1.381, 1.387]	0.796	104
2.5	[1.383, 1.391]	1.161	46	[1.385, 1.389]	0.594	73
2.0	[1.383, 1.389]	0.845	41	[1.386, 1.389]	0.461	54
1.5	[1.383, 1.389]	0.855	36	[1.385, 1.389]	0.589	41
1.0	[1.381, 1.387]	0.770	30	[1.382, 1.389]	0.810	30

Table 5.4: Comparison of performance for different values of  $\beta$  between two choices of p.m.f.

**Remark 5.** For  $\beta = 1$ , the probabilistic representation gives a valid solution of ODE (21) in the domain  $[-l, l]$  provided that the branching diffusion goes extinct almost surely. In addition, according to (6), extinction occurs almost surely if and only if  $l \leq \pi/\sqrt{8} \approx 1.11$ . However, if  $l \geq \cosh^{-1}(\sqrt{2}) \approx 0.88$ , the solution provided by the probabilistic representation takes values in  $[0, 1]$  and thus it does not coincide with  $\sqrt{2}/\cosh(x)$ . In particular, Proposition 1 does not hold since there are at least two solutions to ODE (21) satisfying the same Dirichlet condition in this setting.

$x$	$[p_1, p_2]$	Estimate	99% conf. interval	Std. Dev./Mean	Run time (secs)
0	[0.5, 0.5]	0.9595	[0.9588, 0.9602]	0.2820	260
-0.2	[0.5, 0.5]	0.9613	[0.9607, 0.9620]	0.2732	248
0.2	[0.5, 0.5]	0.9613	[0.9606, 0.9620]	0.2736	250

Table 5.5: Numerical results for the unbiased estimator of probabilistic representation for domain  $(-0.9, 0.9)$  which support Remark 5.

**5.3 Example 3.** Finally, we consider a multidimensional semi-linear PDE

$$\Delta u - 2d(u^3 - u) = 0, \quad (22)$$

with an explicit solution  $u(x) = \tanh(\sum_{i=1}^d x_i)$  where  $x = (x_1, \dots, x_d)$ . In  $d = 2$ , we reformulate the above PDE as

$$\frac{1}{2}\Delta u + \beta \left( -\frac{d}{\beta}u^3 + \frac{\beta + d}{\beta}u - u \right) = 0, \quad \text{in } \mathcal{O} = [-0.5, 0.5]^2,$$

$$u = \tanh, \quad \text{on } \partial\mathcal{O}.$$

In Table 5.6, we illustrate the numerical results for the intuitive choice of p.m.f.  $p_1^* = (\beta + d)/(\beta + 2d)$  and  $p_3^* = d/(\beta + 2d)$  for different values of  $x = 0.2$  and the choice of  $\beta$  which produces the most accurate



results. We observe that our estimator remains accurate however, the variance has increased which is an expected effect due to the increase in dimensionality of the domain.

$x$	Estimate	99% conf. interval	Std. Dev./Mean	Relative error	Run time (secs)
(0,0)	-0.0023	[-0.0109, 0.0051]	—	—	36
(0.1,-0.1)	0.0013	[-0.0056, 0.0081]	—	—	34
(0.1,0.1)	0.1993	[0.1895, 0.2043]	14.56	0.21%	34
(-0.1,-0.1)	-0.1983	[-0.2045, -0.1920]	-12.13	0.46%	34

Table 5.6: Numerical results for the unbiased estimator of probabilistic representation of semi-linear PDE (22) in  $d = 2$  with  $\beta = 2.5$ .

## A Proof of Proposition 4

For the purpose of clarity, we split the proof in two lemmas. In addition, for notational convenience, the proofs are carried out in the one-dimensional setting.

**Lemma 7.** *For any  $\phi$  measurable and bounded, the map  $x \mapsto \mathbb{E}[\int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) ds]$  is continuously differentiable and*

$$\partial_x \mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) ds \right] = \mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) \mathcal{W}_\mathcal{O}(s, x, (W_r)_{r \in [0, s]}) ds \right].$$

*Proof.* The proof consists of four steps. The first three parts essentially follow by repeating the arguments in Gobet [12].

*First step.* Denote

$$u(s, x) := \mathbb{E} [\phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta}].$$

Recall that we have the following representation

$$u(s, x) = \int_{\mathcal{O}} G(s, x; 0, y) \phi(y) dy.$$

where  $G$  is the Green function of PDE (10), see, *e.g.*, Ladyženskaja, Solonnikov and Ural'ceva [21, Thm.4.16.2] or Friedman [10, Thm.3.16]. Further, in view of Theorem 4.16.3 in [21], it is clear that  $u \in C^{1,2}(\mathbb{R}_+^* \times \mathcal{O})$  and thus it satisfies

$$\begin{aligned} \partial_t u - \mathcal{L}u &= 0, & \text{in } \mathbb{R}_+^* \times \mathcal{O}, \\ u(0, \cdot) &= \phi, & \text{on } \mathcal{O} \\ u &= 0, & \text{on } \mathbb{R}_+^* \times \partial\mathcal{O} \end{aligned}$$

In addition,  $u$  and its partial derivatives are bounded in  $[t, T] \times \mathcal{O}$  for any  $0 < t < T$ . Further, under Assumption 4.3, the function  $u$  belongs to  $C^{1,3}(\mathbb{R}_+^* \times \mathcal{O})$  (see, *e.g.*, Friedman [10, Thm.3.10]).

*Second step.* Given  $s > 0$ , let us show next that  $N_t := u'(s - t, \bar{X}_t^x) Y_t^x$  is a martingale on  $[0, \zeta_s]$ , or equivalently,  $(N_{\zeta_s \wedge t})_{t \in [0, s]}$  is a martingale. Using Itô's formula, we obtain

$$N_{\zeta_s \wedge t} = u'(s, x) + \int_0^{\zeta_s \wedge t} (\sigma'(\bar{X}_r^x) u'(s - r, \bar{X}_r^x) + \sigma(\bar{X}_r^x) u''(s - r, \bar{X}_r^x)) Y_r^x dW_r.$$

Further, in view of Lemma 3, there exists  $s' < s$  such that  $\zeta_s \leq s'$ . The maps  $u'$  and  $u''$  being bounded in  $[s - s', s] \times \mathcal{O}$ , we conclude that  $(N_{\zeta_s \wedge t})_{t \in [0, s]}$  is a martingale.

*Third step.* We are now in a position to complete the proof. Indeed, Itô's formula yields for all  $t < s$ ,

$$u(s - \zeta_s, \bar{X}_{\zeta_s}^x) = u(s, x) + \int_0^{\zeta_s} \sigma(\bar{X}_r^x) u'(s - r, \bar{X}_r^x) dW_r.$$

Using successively Markov property and Itô isometry, we obtain

$$\begin{aligned} \mathbb{E} \left[ \phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta} \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r \right] &= \mathbb{E} \left[ u(s - \zeta_s, \bar{X}_{\zeta_s}^x) \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r \right] \\ &= \mathbb{E} \left[ \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) N_r dr \right]. \end{aligned}$$

Using successively  $N_{\zeta_s \wedge r} = \mathbb{E}[N_{\zeta_s} | \mathcal{F}_r]$  and  $\int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) dr = 1$ , we deduce that

$$\mathbb{E} \left[ \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) N_r dr \right] = \mathbb{E} \left[ N_{\zeta_s} \int_0^{\zeta_s} \theta_s(r, \bar{X}_r^x) dr \right] = \mathbb{E}[N_{\zeta_s}] = u'(s, x).$$

*Fourth step.* By Fubini's theorem, we have

$$\mathbb{E} \left[ \int_0^\eta e^{-\beta s} \phi(\bar{X}_s^x) ds \right] = \int_0^{+\infty} e^{-\beta s} \mathbb{E} \left[ \phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta} \right] ds. \quad (23)$$

In addition, from the former steps, it holds

$$\partial_x \mathbb{E} \left[ \phi(\bar{X}_s^x) \mathbf{1}_{s \leq \eta} \right] = \mathbb{E} \left[ \phi(\bar{X}_s^x) \mathcal{W}(s, x, (W_r)_{r \in [0, s]}) \mathbf{1}_{s \leq \eta} \right]$$

Finally, in view of the estimates in Proposition 4, we can apply differentiation under the integral sign in (23). This completes the proof.  $\square$

**Lemma 8.** *For any  $\bar{\phi} \in \mathcal{C}^{1, \alpha}(\bar{\mathcal{O}})$ , the map  $x \mapsto \mathbb{E}[e^{-\beta \eta} \bar{\phi}(\bar{X}_\eta^x)]$  is continuously differentiable and*

$$\partial_x \mathbb{E} [e^{-\beta \eta} \bar{\phi}(\bar{X}_\eta^x)] = \mathbb{E} [e^{-\beta \eta} \bar{\phi}(\bar{X}_\eta^x) \mathcal{W}_{\partial \mathcal{O}}(x, (W_r)_{r \in [0, \eta]})].$$

*Proof.* This proof is similar to the proof of Lemma 7 and we split it in three parts.

*First step.* Denote

$$u(x) := \mathbb{E} [e^{-\beta \eta} \bar{\phi}(\bar{X}_\eta^x)].$$

Recall that  $u \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$  satisfies the following PDE:

$$\begin{aligned} \mathcal{L}u - \beta u &= 0, & \text{in } \mathcal{O}, \\ u &= \bar{\phi}, & \text{on } \partial \mathcal{O} \end{aligned}$$

In addition, since  $\bar{\phi} \in \mathcal{C}^{1, \alpha}(\bar{\mathcal{O}})$  by assumption, it is known that  $u \in \mathcal{C}^{1, \alpha}(\bar{\mathcal{O}})$ , see, *e.g.*, Gilbarg and Trudinger [11, Thm.8.34]. Further, under Assumption 4.3, the function  $u$  belongs to  $\mathcal{C}^3(\mathcal{O})$  (see, *e.g.*, Gilbarg and Trudinger [11, Thm.6.17]).

*Second step.* Let us show next that  $N_s := e^{-\beta s} u'(\bar{X}_s^x) Y_s^x$  is a martingale on  $[0, \eta]$ , or equivalently,  $(N_{\eta \wedge s})_{s \geq 0}$  is a martingale. Using Itô's formula, we obtain

$$N_{\eta \wedge s} = u'(x) + \int_0^{\eta \wedge s} e^{-\beta r} (\sigma'(\bar{X}_r^x) u'(\bar{X}_r^x) + \sigma(\bar{X}_r^x) u''(\bar{X}_r^x)) Y_r^x dW_r.$$

Thus  $(N_{\eta \wedge s})_{s \geq 0}$  is a local martingale. To conclude, it remains to observe that for any  $s \geq 0$ ,

$$\mathbb{E} \left[ \sup_{0 \leq r \leq s} \{N_{\eta \wedge r}\} \right] < +\infty.$$

since  $u'$  is bounded as a continuous function in  $\bar{\mathcal{O}}$  and  $\sup_{0 \leq r \leq s} \{|Y_r^x|\}$  admits finite moment of any order.

*Third step.* We are now in a position to complete the proof. Indeed, Itô's formula yields for all  $s \geq 0$ ,

$$\bar{\phi}(\bar{X}_\eta^x) = u(x) + \int_0^\eta \sigma(\bar{X}_r^x) u'(\bar{X}_r^x) dW_r.$$

Using Itô isometry, we obtain

$$\mathbb{E} \left[ \bar{\phi}(\bar{X}_\eta^x) \int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) \sigma^{-1}(\bar{X}_r^x) Y_r^x dW_r \right] = \mathbb{E} \left[ \int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) N_r dr \right].$$

Using successively  $N_{\zeta_T \wedge r} = \mathbb{E}[N_{\zeta_T} | \mathcal{F}_r]$  and  $\int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) dr = 1$ , we deduce that

$$\mathbb{E} \left[ \int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) N_r dr \right] = \mathbb{E} \left[ N_{\zeta_T} \int_0^{\zeta_T} \theta_T(r, \bar{X}_r^x) dr \right] = \mathbb{E}[N_{\zeta_T}] = u'(x).$$

□

## References

- [1] K. B. Athreya and P. E. Ney. *Branching processes*, volume 196. Springer Science & Business Media, 2012.
- [2] M. Badiale and E. Serra. *Semilinear elliptic equations for beginners*. Universitext. Springer, London, 2011. ISBN 978-0-85729-226-1.
- [3] V. Bally and G. Pagès. Error analysis of the optimal quantization algorithm for obstacle problems. *Stochastic Process. Appl.*, 106(1):1–40, 2003.
- [4] M. Bossy, N. Champagnat, H. Leman, S. Maire, L. Violeau, and M. Yvinec. Monte Carlo methods for linear and non-linear Poisson-Boltzmann equation. *CEMRACS 2013—modelling and simulation of complex systems: stochastic and deterministic approaches, ESAIM Proc. Surveys*, 48:420–446, 2015.
- [5] B. Bouchard, I. Ekland, and N. Touzi. On the Malliavin approach to Monte Carlo approximation of conditional expectations. *Finance and Stochastics*, 8(1):45–71, 2004.
- [6] B. Bouchard, X. Tan, and Y. Zou. Numerical approximation of BSDEs using local polynomial drivers and branching processes. *ArXiv e-prints*, Dec. 2016.
- [7] P. Coolen-Schrijner and E. van Doorn. Quasi-stationary distributions for a class of discrete-time markov chains. *Memorandum / Department of Applied Mathematics*, (1737), 2004.
- [8] D. J. Daley. Quasi-stationary behaviour of a left-continuous random walk. *The Annals of Mathematical Statistics*, 40(2):532–539, 1969.
- [9] F. Delarue. Estimates of the solutions of a system of quasi-linear PDEs. A probabilistic scheme. In *Séminaire de Probabilités XXXVII*, pages 290–332. Springer, 2003.
- [10] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- [11] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.
- [12] E. Gobet. Revisiting the Greeks for European and American options. In *Proceedings of the International Symposium on Stochastic Processes and Mathematical Finance, Ritsumeikan University, Kusatsu, Japan*, pages 53–71, 2004.
- [13] E. Gobet and S. Menozzi. *Discrete Sampling of Functionals of Ito Processes*, pages 355–374. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.
- [14] P. Henry-Labordère. Cutting CVA’s complexity. *Risk*, 25(7):67, 2012.
- [15] P. Henry-Labordère, X. Tan, and N. Touzi. A numerical algorithm for a class of BSDEs via the branching process. *Stochastic Processes and their Applications*, 124(2):1112–1140, 2014.
- [16] P. Henry-Labordère, N. Touzi, and X. Tan. Exact simulation of multi-dimensional stochastic differential equations. *Available at SSRN*, 2015.
- [17] P. Henry-Labordère, N. Oudjane, X. Tan, N. Touzi, and X. Warin. Branching diffusion representation of semilinear PDEs and Monte Carlo approximation. *arXiv preprint arXiv:1603.01727*, 2016.

- [18] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. I. *J. Math. Kyoto Univ.*, 8: 233–278, 1968. ISSN 0023-608X.
- [19] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. II. *J. Math. Kyoto Univ.*, 8: 365–410, 1968. ISSN 0023-608X.
- [20] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. III. *J. Math. Kyoto Univ.*, 9: 95–160, 1969. ISSN 0023-608X.
- [21] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
- [22] A. Lejay. exitbm: a library for simulating Brownian motion's exit times and positions from simple domains. Technical Report RR-7523, Feb. 2011. URL [HTTPS://HAL.INRIA.FR/INRIA-00561409](https://hal.inria.fr/inria-00561409).
- [23] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Communications on Pure and Applied Mathematics*, 28(3):323–331, 1975.
- [24] É. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In *Stochastic analysis and related topics, VI (Geilo, 1996)*, volume 42 of *Progr. Probab.*, pages 79–127. Birkhäuser Boston, Boston, MA, 1998.
- [25] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.
- [26] B. A. Sevast'janov. The extinction conditions for branching processes with diffusion. *Akademiya Nauk SSSR. Teoriya Veroyatnostei i ee Primeneniya*, 6:276–286, 1961.
- [27] A. V. Skorohod. Branching diffusion processes. *Akademiya Nauk SSSR. Teoriya Veroyatnostei i ee Primeneniya*, 9:492–497, 1964.
- [28] A. Thalmaier. On the differentiation of heat semigroups and Poisson integrals. *Stochastics and Stochastics Reports*, 61(3-4):297–321, 1997.
- [29] A. Thalmaier and F.-Y. Wang. Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *Journal of Functional Analysis*, 155(1):109–124, 1998.
- [30] S. Watanabe. On the branching process for Brownian particles with an absorbing boundary. *Journal of Mathematics of Kyoto University*, 4:385–398, 1965.
- [31] J. Zhang. A numerical scheme for BSDEs. *The Annals of Applied Probability*, 14(1):459–488, 2004.